# A New Variational Approach for Limited Angle Tomography

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Microscopy Collaborators: Rowan Leary, Sean Collins, Paul Midgley

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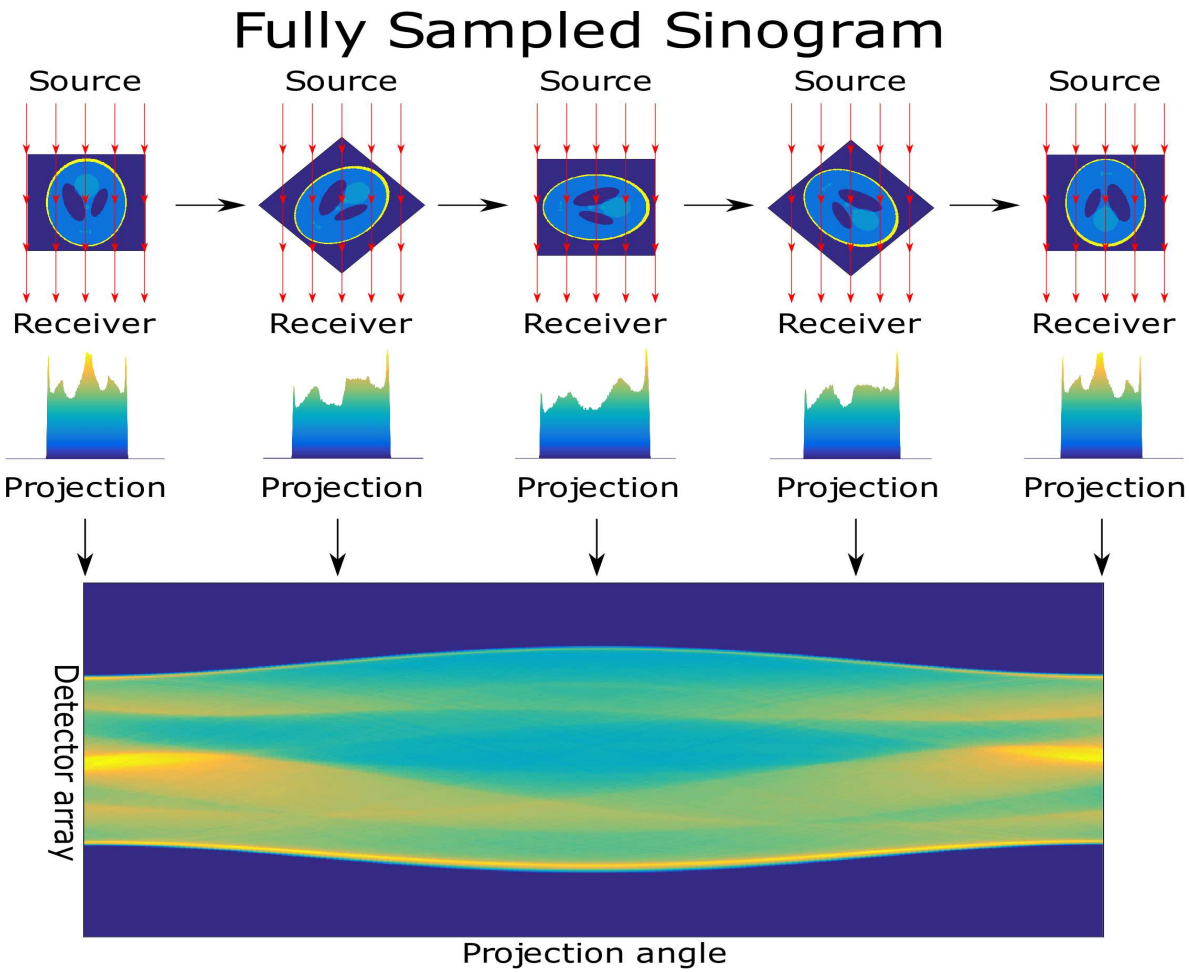
# Outline


1. Problem Motivation
2. Proposed Sparsity Model
3. Non-Convex and Non-Differentiable Optimisation
4. Numerical Experiments

# Data Acquisition

X-Ray forward model with noise:

$$\underbrace{\mathcal{R}}_{\text{full transform}} u + \underbrace{\eta}_{\text{noise}} = \underbrace{b}_{\text{data}}$$



# Data Acquisition

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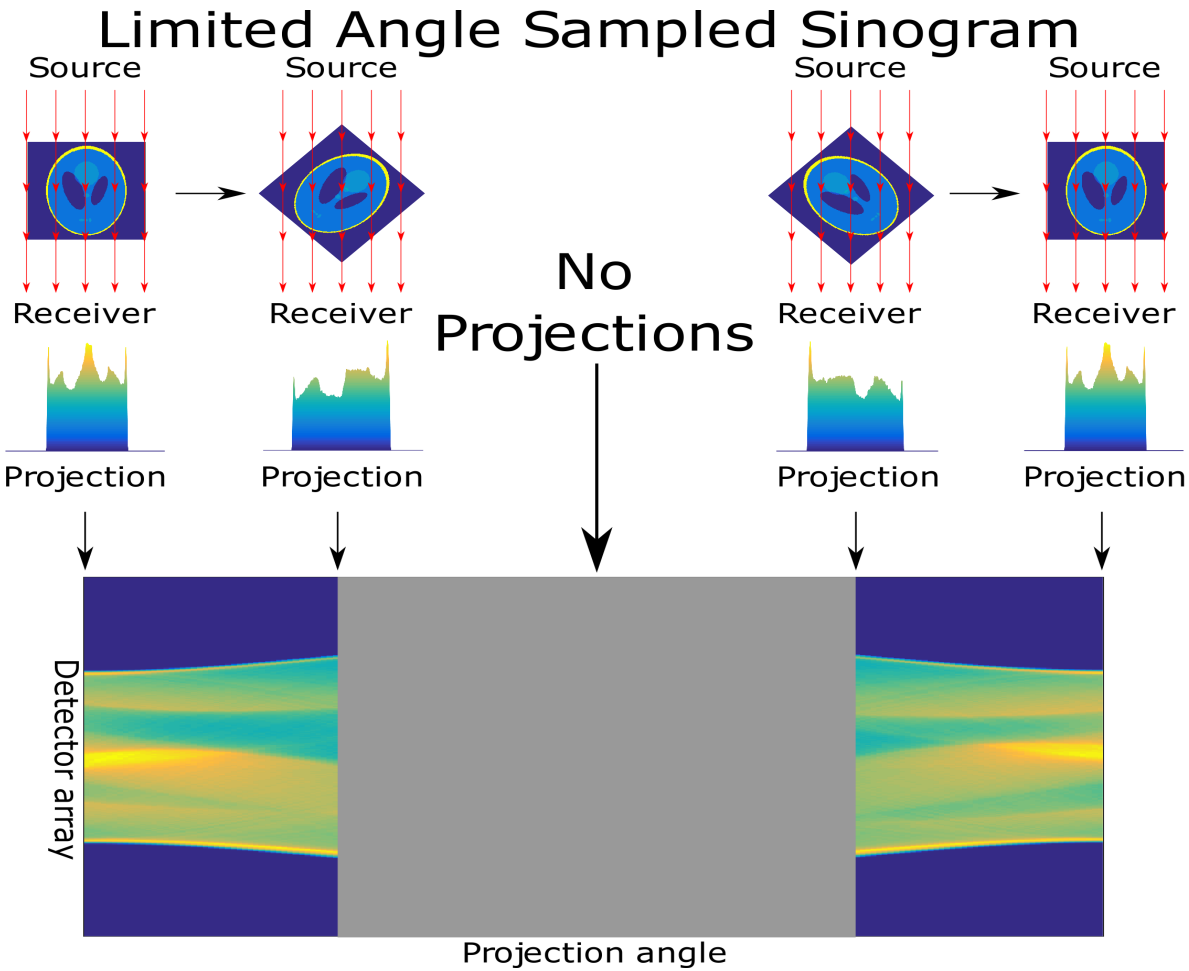
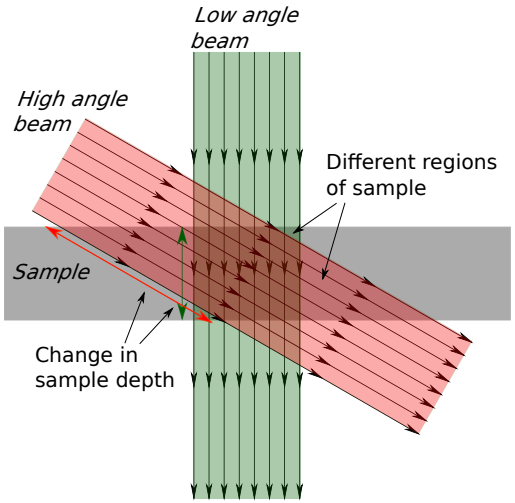
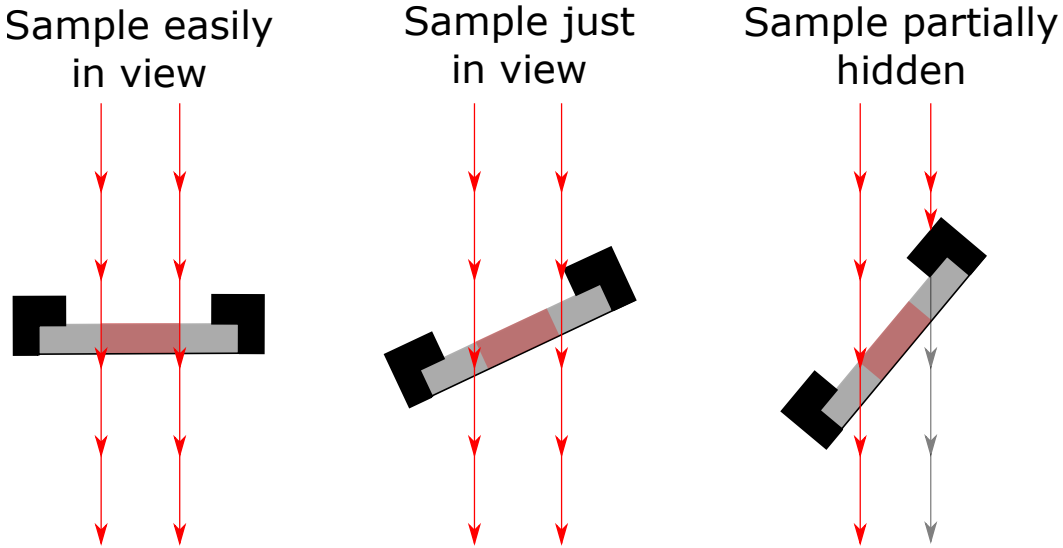
$$\underbrace{S}_{\text{subsample}} \quad \underbrace{\mathcal{R}}_{\text{full transform}} \quad u + \underbrace{\eta}_{\text{noise}} = \underbrace{b}_{\text{data}}$$



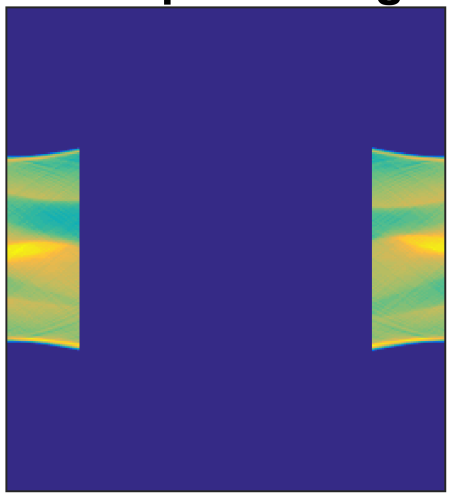
# Physical Motivation



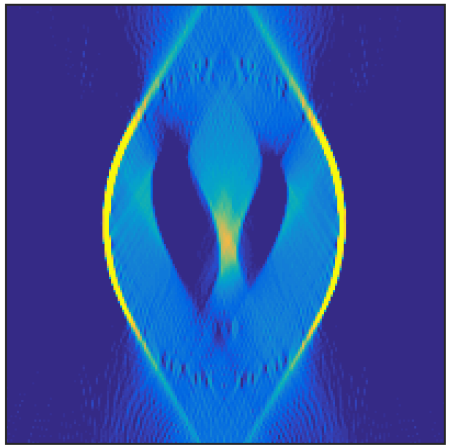



# Impact on Reconstructions

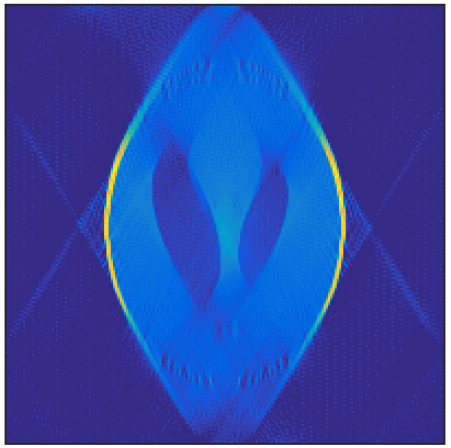
**Sub-sampled Sinogram**



**FBP reconstruction**



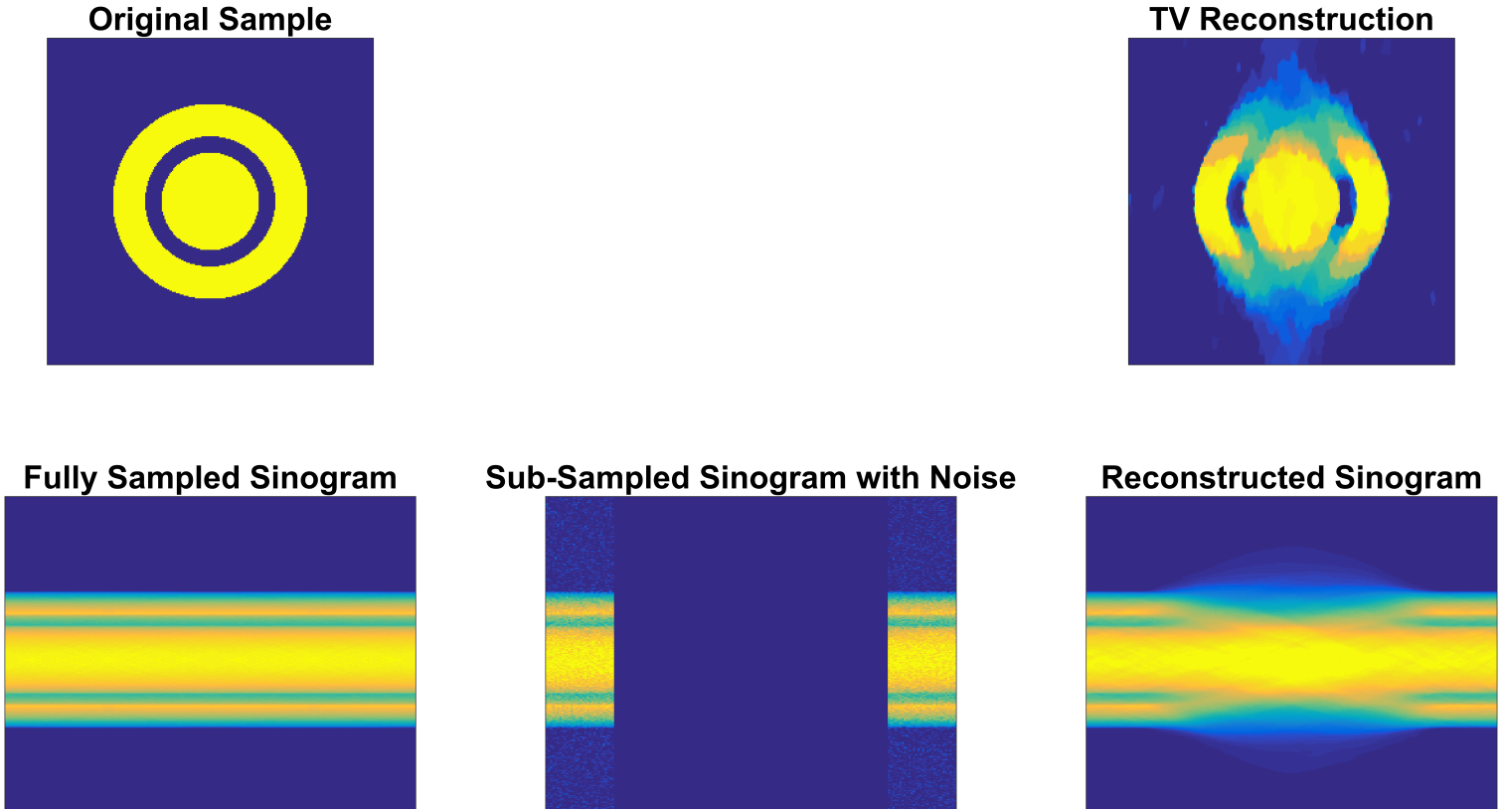
**SIRT reconstruction**



# A Simpler Example

Standard Model: Total Variational Reconstruction

$$\text{reconstruction} = \underset{u}{\operatorname{argmin}} \frac{1}{2} \|S\mathcal{R}u - b\|_2^2 + \lambda \|\nabla u\|_{2,1}$$




*Compressed sensing in electron tomography*, Leary, Saghi, Midgley, Holland 2013

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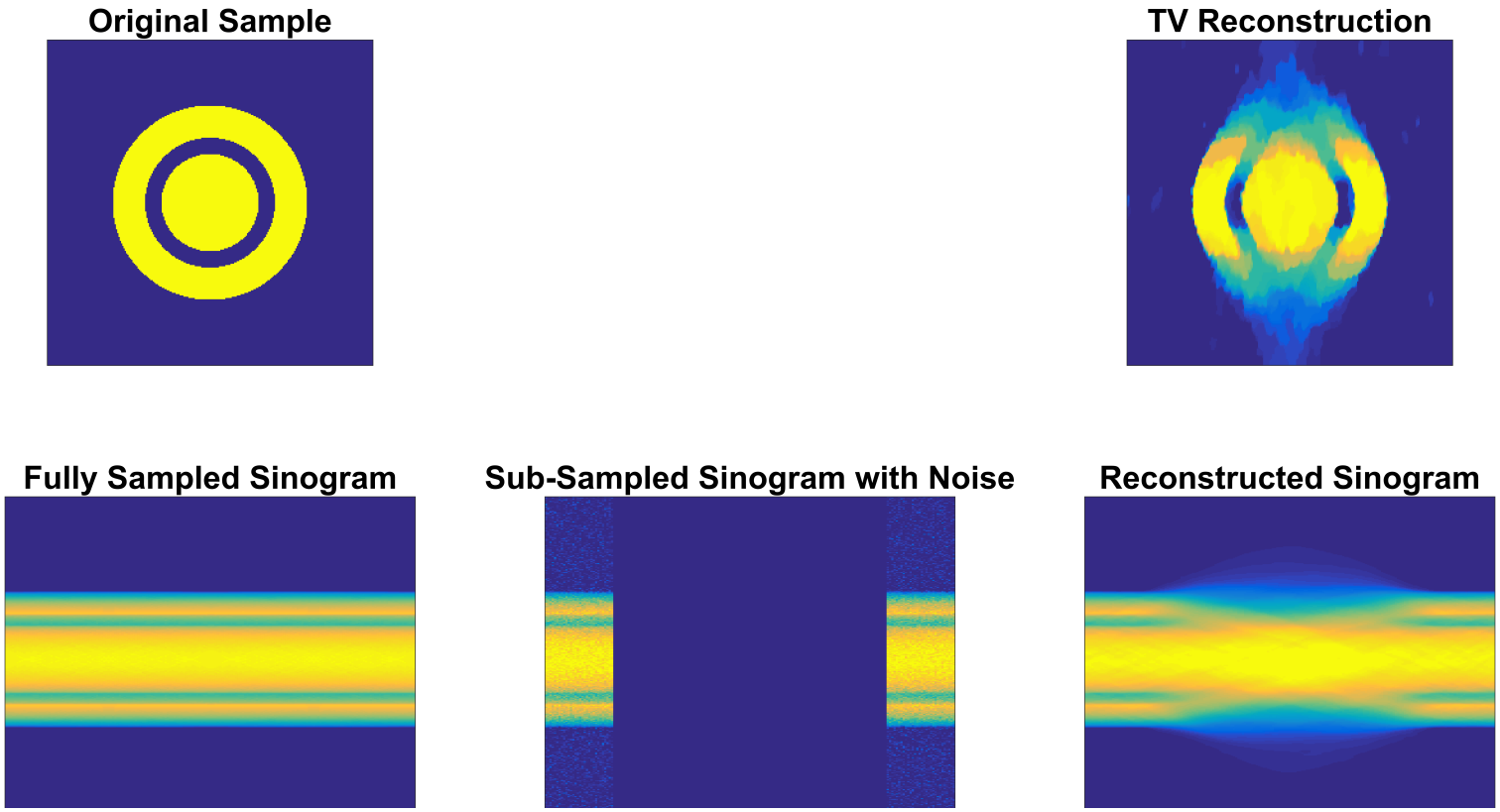


The solution: global regularisation in data space

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*Compressed sensing in electron tomography*, Leary, Saghi, Midgley, Holland 2013

# Anisotropic Total Variation

Method from inpainting literature:

$$\text{reconstruction} = \underset{v}{\operatorname{argmin}} \frac{1}{2} \|Sv - b\|_2^2 + \lambda \|A\nabla v\|_{2,1}$$

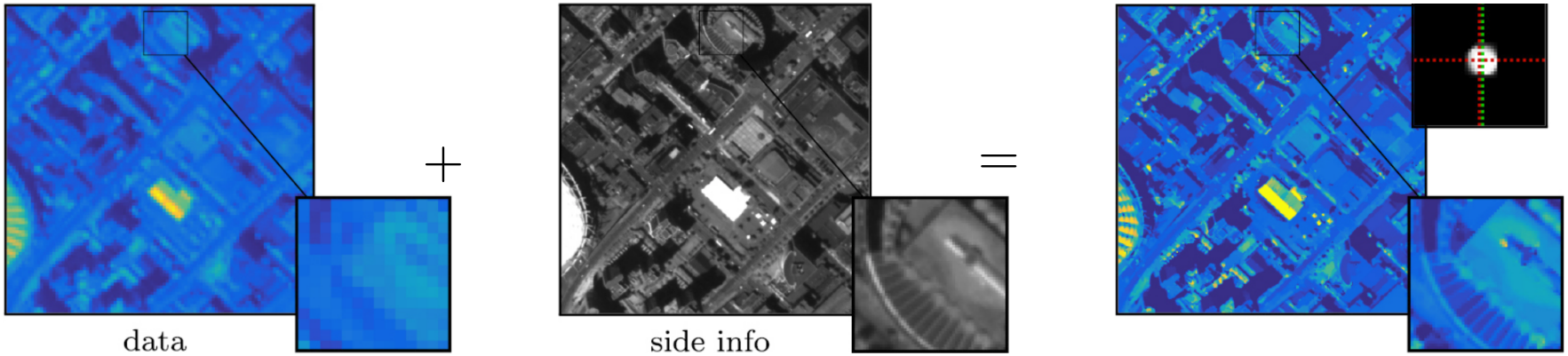


Figure adapted from *Blind image fusion for hyperspectral imaging with the directional total variation*, Bungert, Coomes, Ehrhardt, Rasch, Reisenhofer, Schönlieb 2018

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*Anisotropic Diffusion in Image Processing*, Weickert 1998

*A flexible space-variant anisotropic regularisation for image restoration with automated parameter selection*, Calatroni, Lanza, Pragliola, Sgallari 2019

# Our Model

Energy Functional:

$$E(u,v) = \underbrace{\frac{1}{2}\|Sv - b\|_2^2 + \alpha\|A\nabla v\|_{2,1}}_{\text{inpainting problem}}$$
$$+ \underbrace{\frac{1}{2}\|\mathcal{R}u - v\|_\beta^2 + \gamma\|\nabla u\|_{2,1} + \chi_{u\geq 0}}_{\text{fully sampled reconstruction}}$$

Reconstruction Method:

$$\text{reconstruction} = \underset{u,v}{\operatorname{argmin}}\, E(u,v)$$

where $A$ is an anisotropic diffusion tensor.

# Our Model

Energy Functional:

$$E(u,v) = \underbrace{\frac{1}{2}\|Sv - b\|_2^2 + \alpha\|A(\mathcal{R}u)\nabla v\|_{2,1}}_{\text{inpainting problem}}$$
$$+ \underbrace{\frac{1}{2}\|\mathcal{R}u - v\|_\beta^2 + \gamma\|\nabla u\|_{2,1} + \chi_{u\geq 0}}_{\text{fully sampled reconstruction}}$$

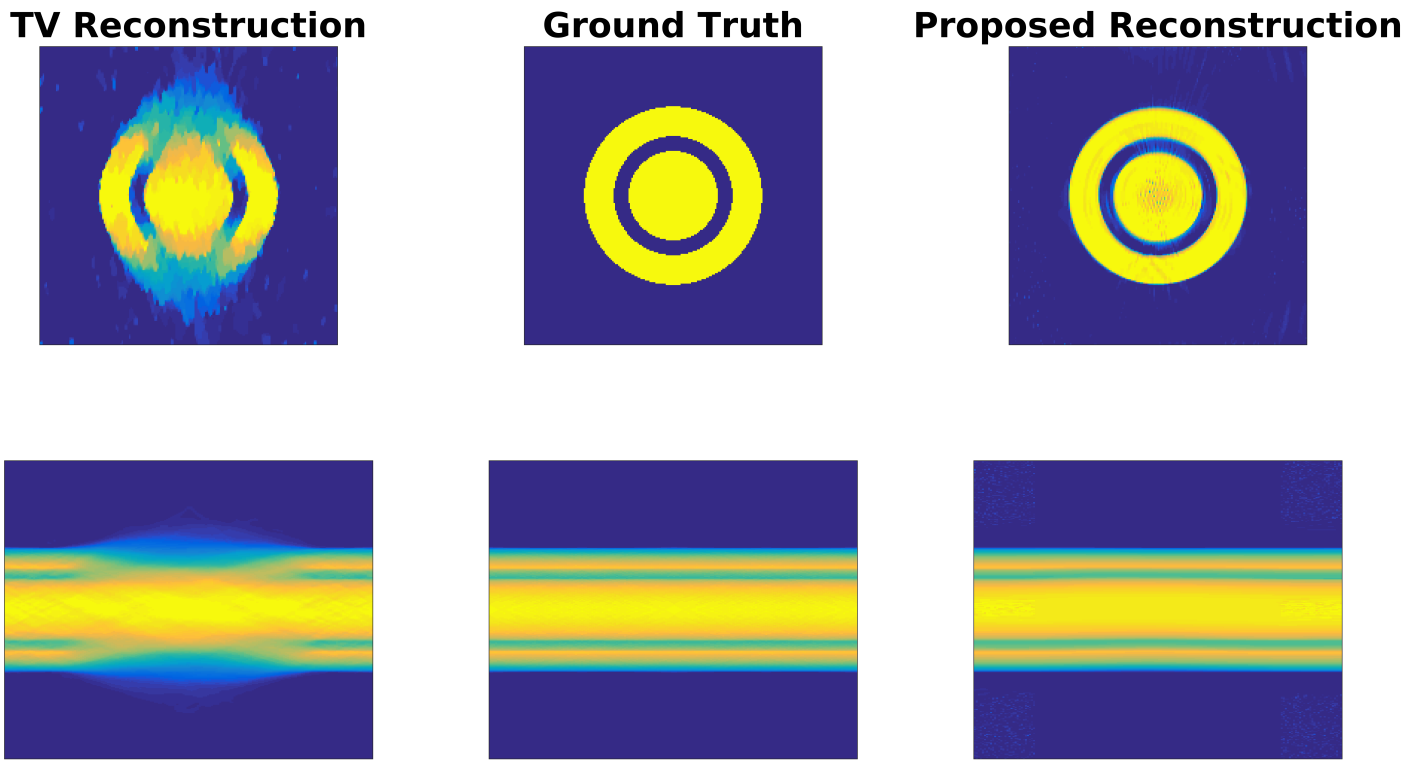
Problem:

$$A = A(\text{reconstruction}) \rightsquigarrow A = A(\mathcal{R}u)$$

**Theorem**

*For suitable choices of hyperparameters, $A \in C^\infty$ and $E$ is weakly lower semi-continuous.*

# Sanity Check




It is a hard non-convex/non-smooth optimization problem but it does add the right sort of information.

# Literature Review

| Reference | Structure | Complexity | Intepretability |
|---|---|---|---|
| Dong, Li, Shen 2013 | Wavelet | Convex | High |
| Current talk 2019 | Anisotropy | near-Convex | Medium |
| Bubba, Kutyniok, et. al. 2018 | Learned | non-convex | Low |

*X-ray CT image reconstruction via wavelet frame based regularization and Radon domain inpainting*, Dong, Li, Shen 2013
*Learning the invisible: a hybrid deep learning-shearlet framework for limited angle computed tomography*, Bubba, Kutyniok, Lassas, März, Samek, Siltanen, Srinivasan 2018

1 Problem Motivation

2 Proposed Sparsity Model

3 Non-Convex and Non-Differentiable Optimisation

4 Numerical Experiments

# Can we avoid Non-Convex/Non-Differentiable?

Generalization of the model:

$$E(u, v) = f(u, v) + \|A(u)v\|_1$$

where $f$ is simple, jointly-convex.

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- Non-Convex/-Differentiable $\rightsquigarrow$ not optimizable directly
- Mantra: simplify$\rightarrow$penalize$\rightarrow$optimize$\rightarrow$repeat...
- Our solution, (bi-)convexify:

$$A(u)v \approx A(u_0)v + \nabla A(u_0)(u - u_0)v$$

is a bilinear.

# The Alternative

$$u_{n+1} = \underset{u}{\operatorname{argmin}}\, f(u, v_n) + \|[A(u_n) + \nabla A(u_n)(u - u_n)]v_n\|_1 + \tau \|u - u_n\|_2^2$$

$$v_{n+1} = \underset{v}{\operatorname{argmin}}\, f(u_{n+1}, v) + \|A(u_{n+1})v\|_1 + \|v - v_n\|_2^2$$

*Error bounds, quadratic growth, and linear convergence of proximal methods*, Drusvyatskiy and Lewis 2016

*Non-smooth Non-convex Bregman Minimization: Unification and new Algorithms*, Ochs, Fadili, and Brox 2017

# Convergence Results

**Theorem**

1. *In general Banach spaces we have a monotone decrease property*
$$E(u_{n+1}, v_{n+1}) \leq E(u_n, v_n)$$
$$\sum^{N} \|u_{n+1} - u_n\|_2^2 + \|v_{n+1} - v_n\|_2^2 \leq E(u_0, v_0) - E(u_{N+1}, v_{N+1})$$
2. *In finite dimensions, a subsequence must converge*
3. *Any limit point must be critical in u and critical in v*

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*Proximal alternating linearized minimization for nonconvex and nonsmooth problems*, Bolte, Sabach, Teboulle 2013
*Non-smooth Non-convex Bregman Minimization: Unification and new Algorithms*, Ochs, Fadili, Brox 2017

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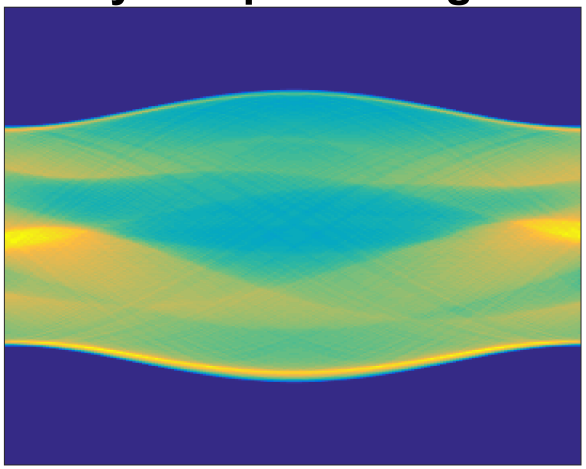
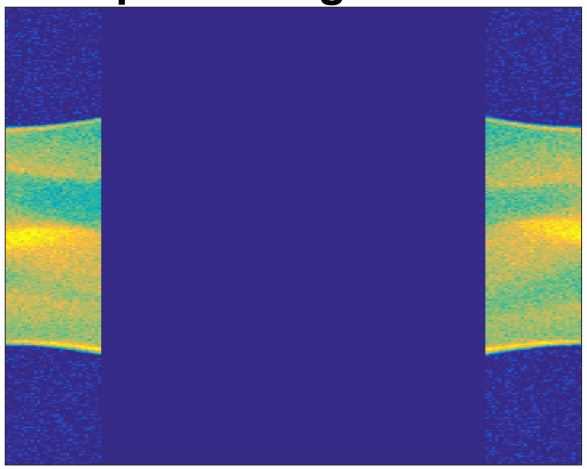
# Shepp-Logan Phantom Example

**Original Sample**

**Fully Sampled Sinogram**

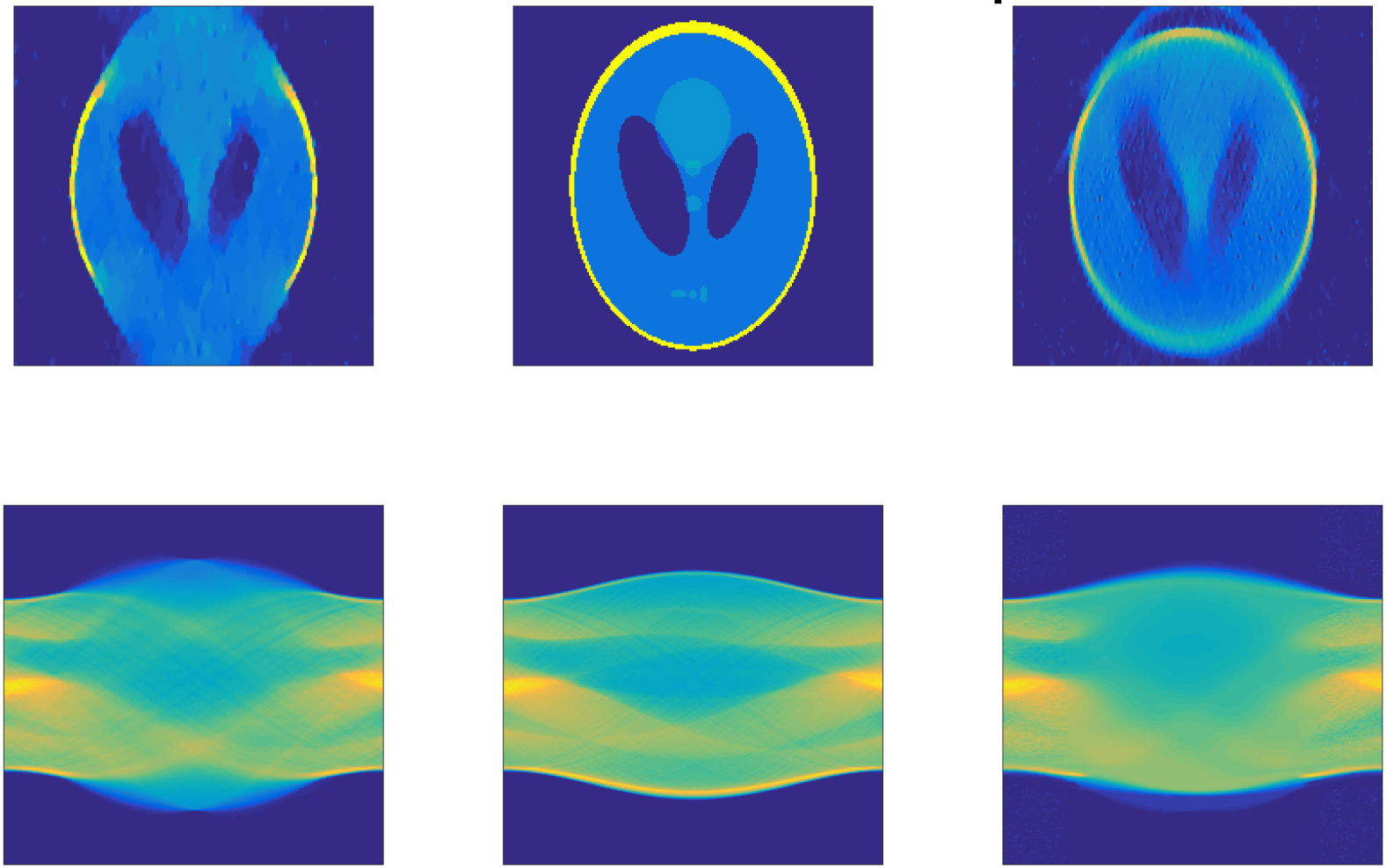


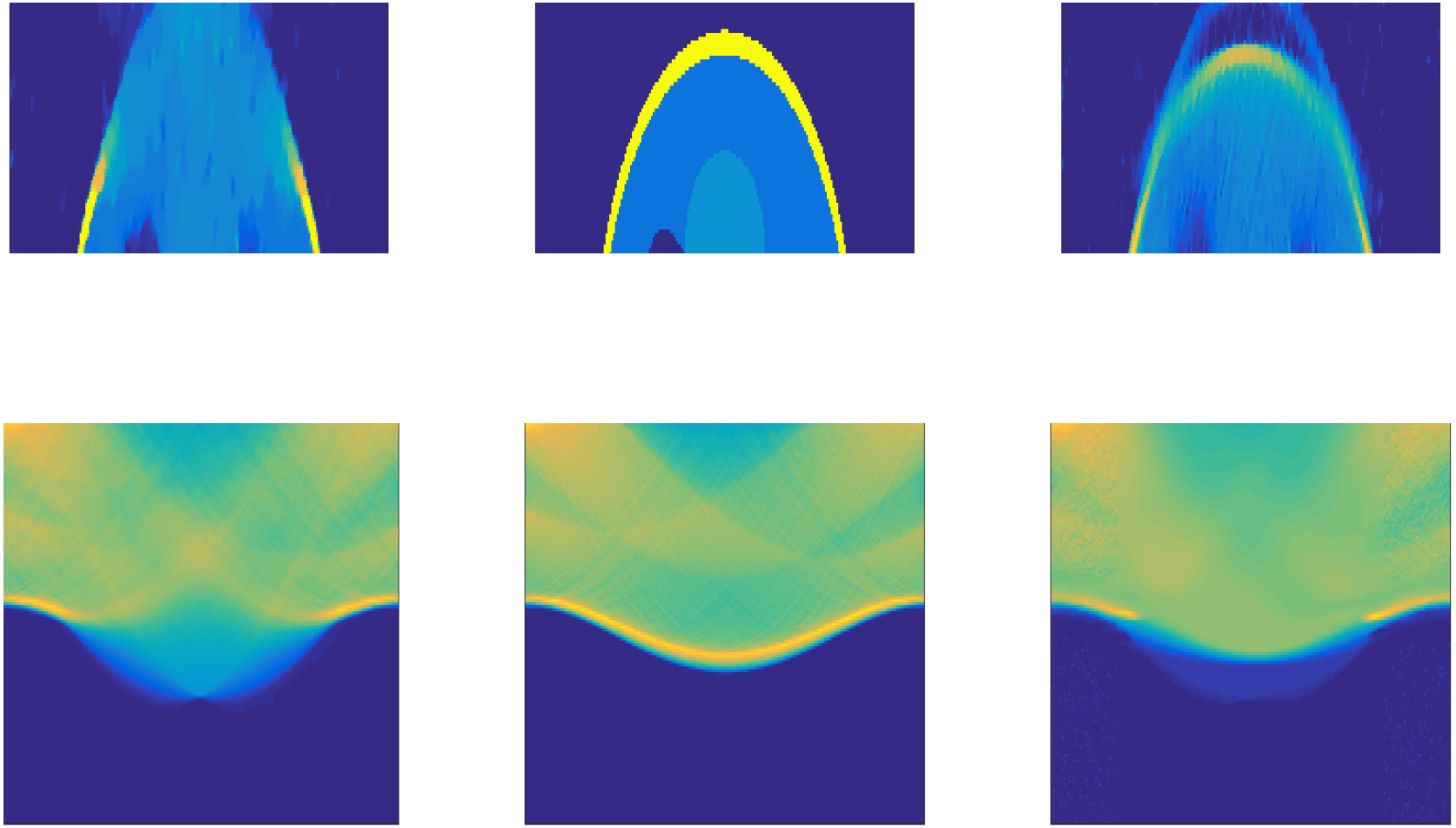
**Sub-Sampled Sinogram with Noise**

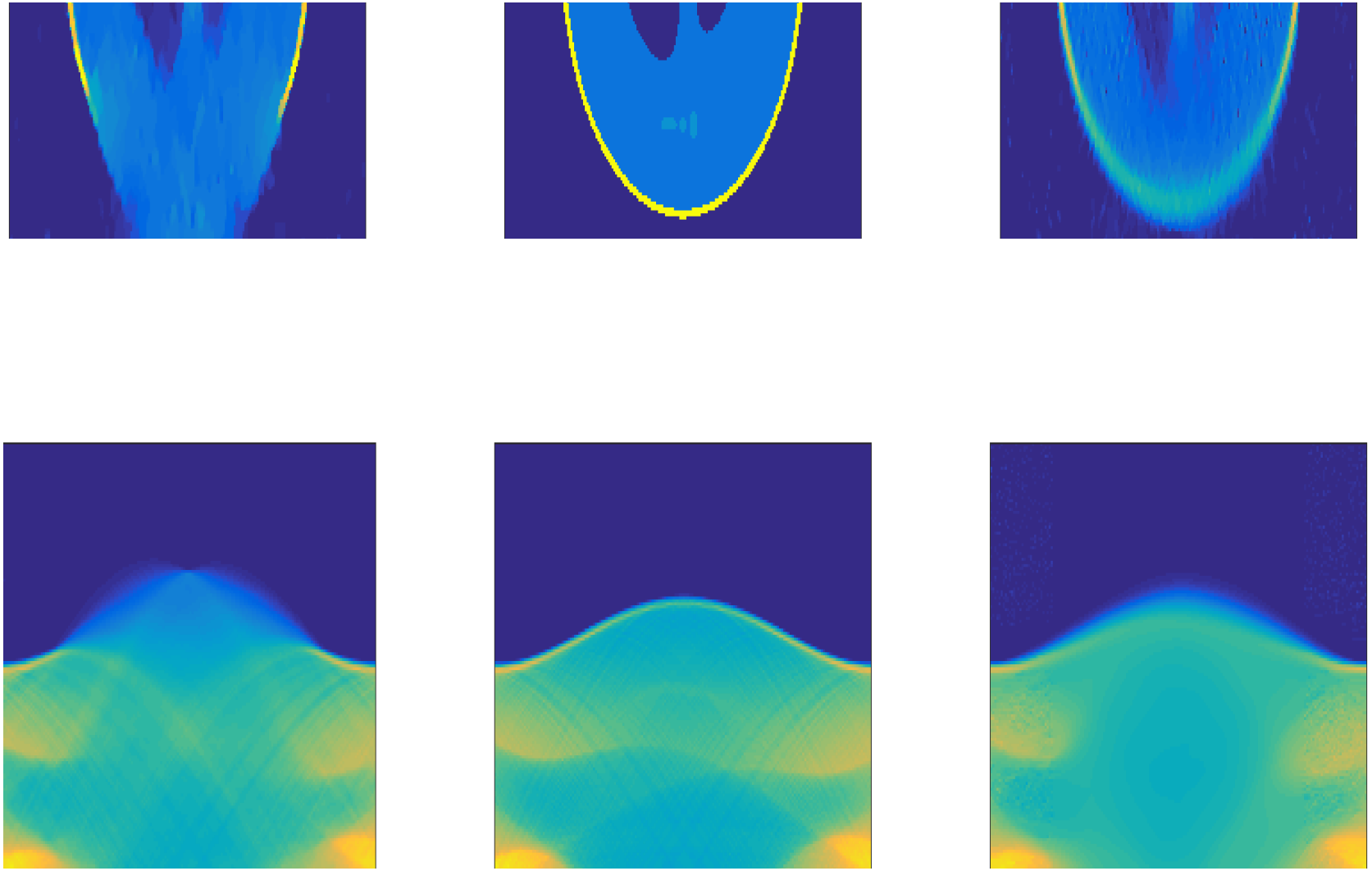


# Shepp-Logan Phantom Example

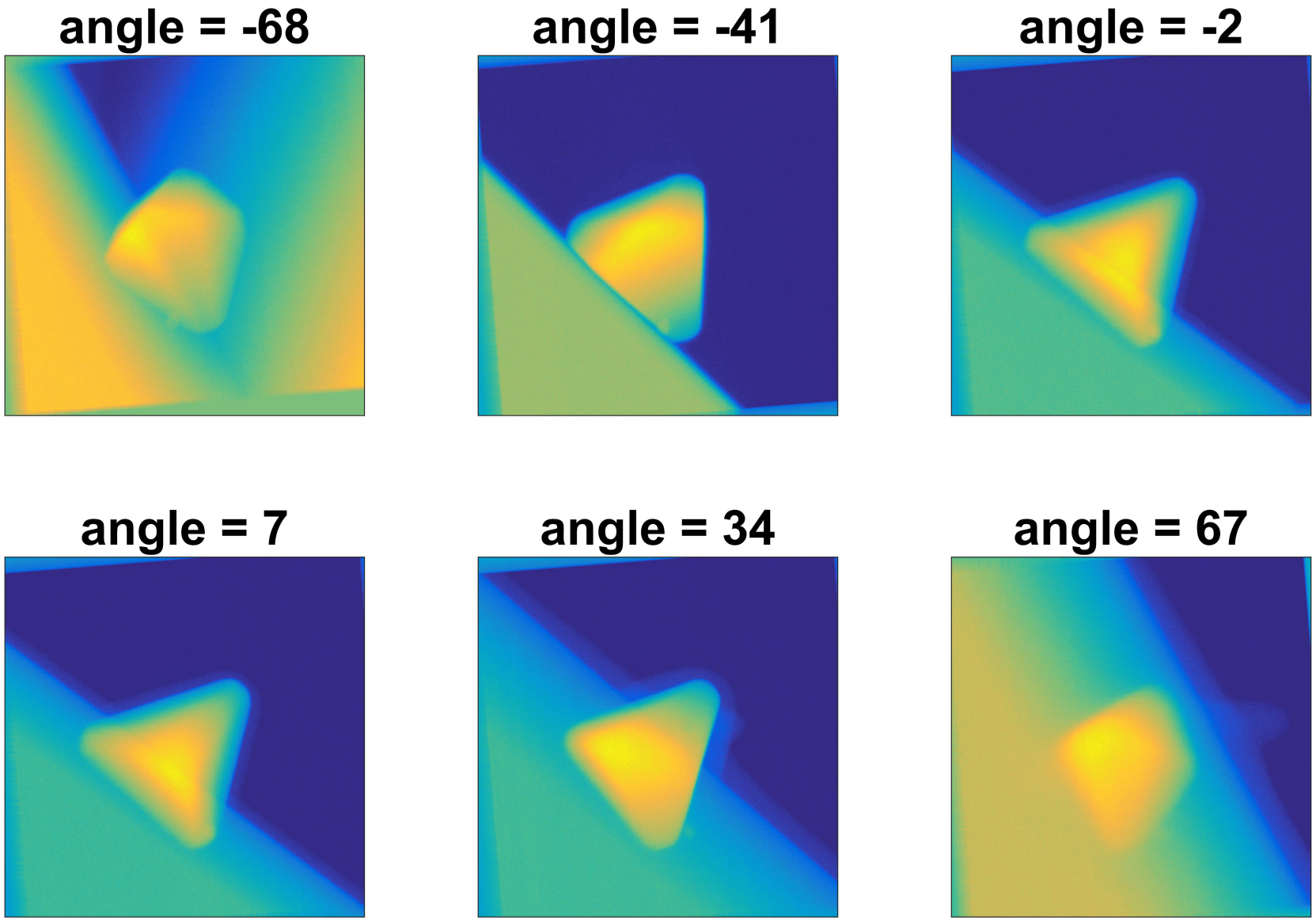
**TV Reconstruction** **Ground Truth** **Proposed Reconstruction**





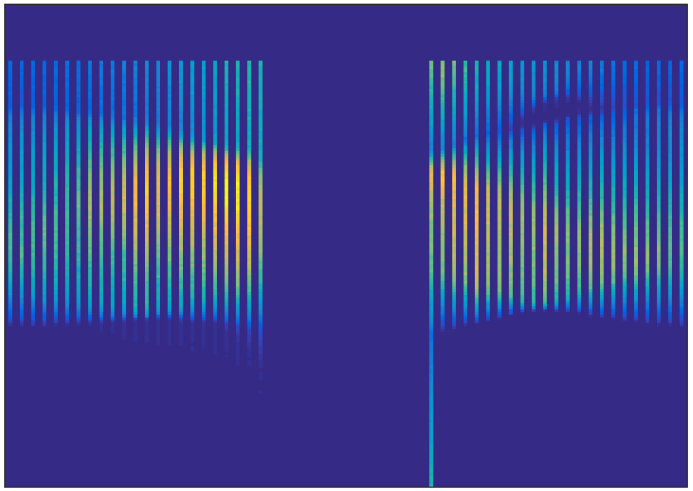


# Experimental Example

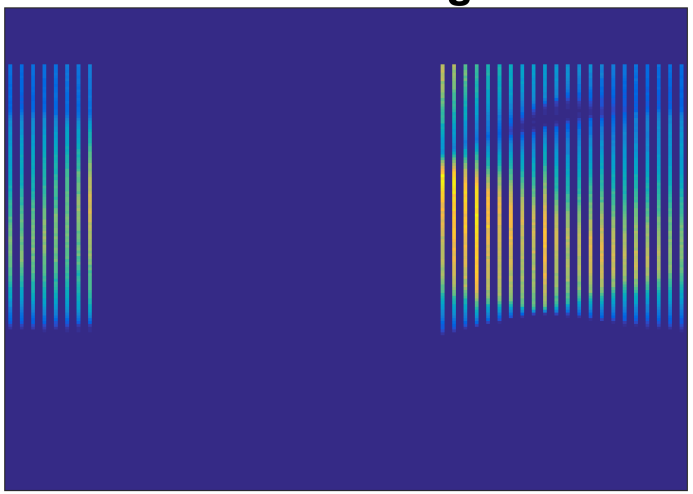


# Experimental Example

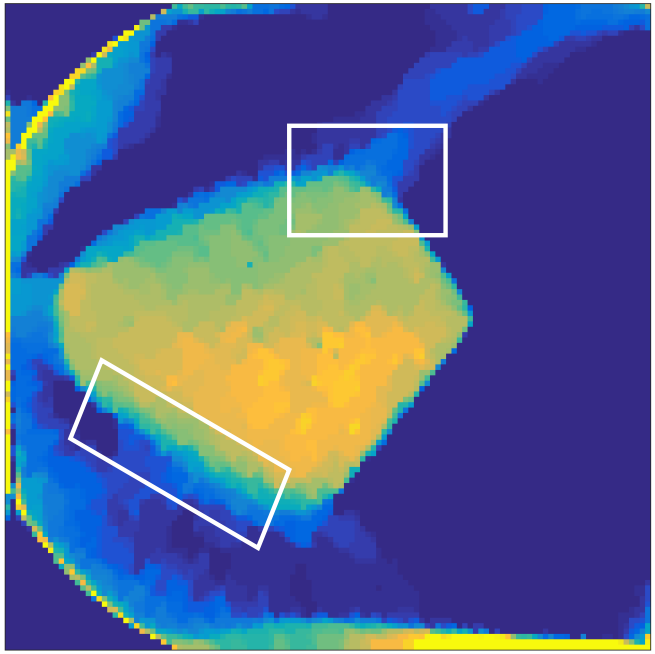
**Pre-Processed Data, Full Range**



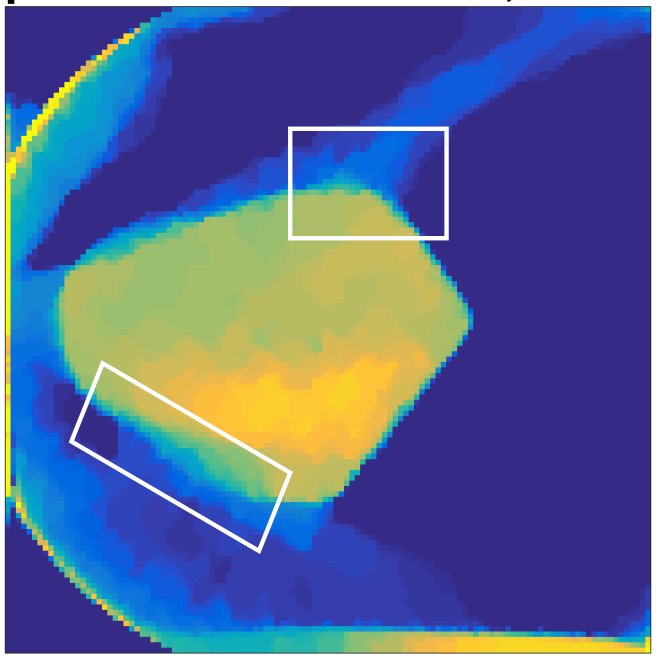
**Pre-Processed Data, Limited Range**



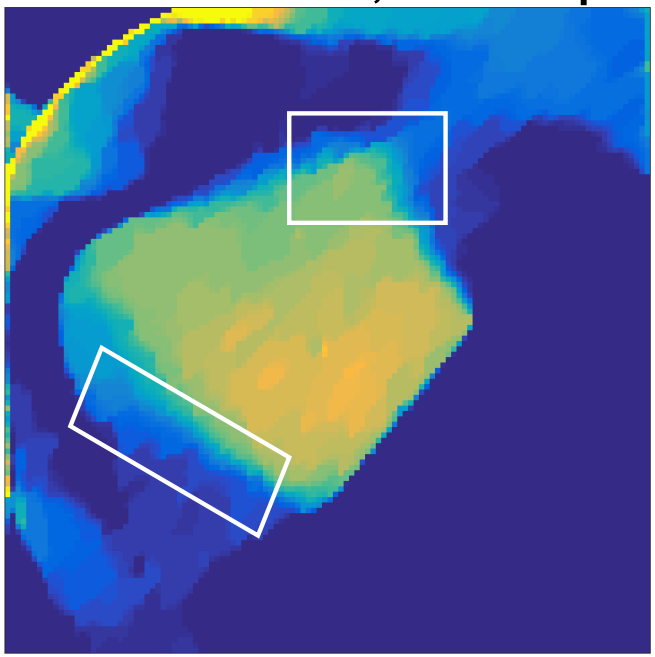
**TV Reconstruction, Full Data**



**Proposed Reconstruction, Full Data**



**TV Reconstruction, Sub-sampled**



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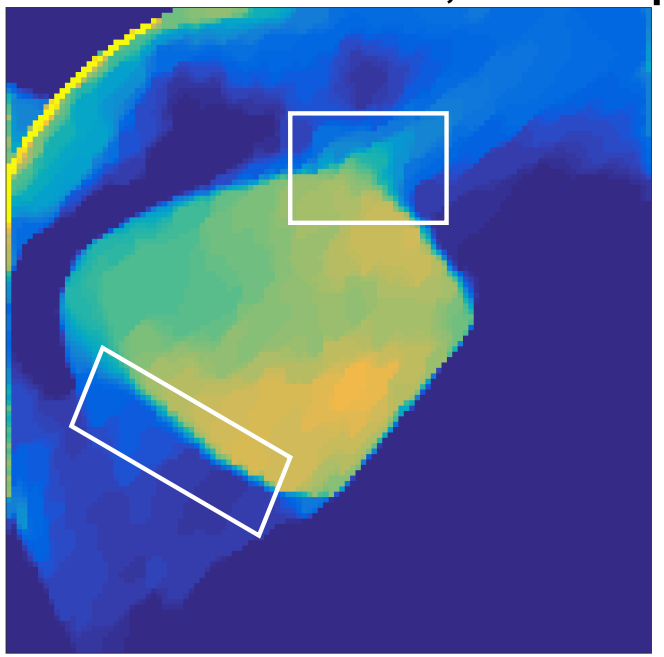
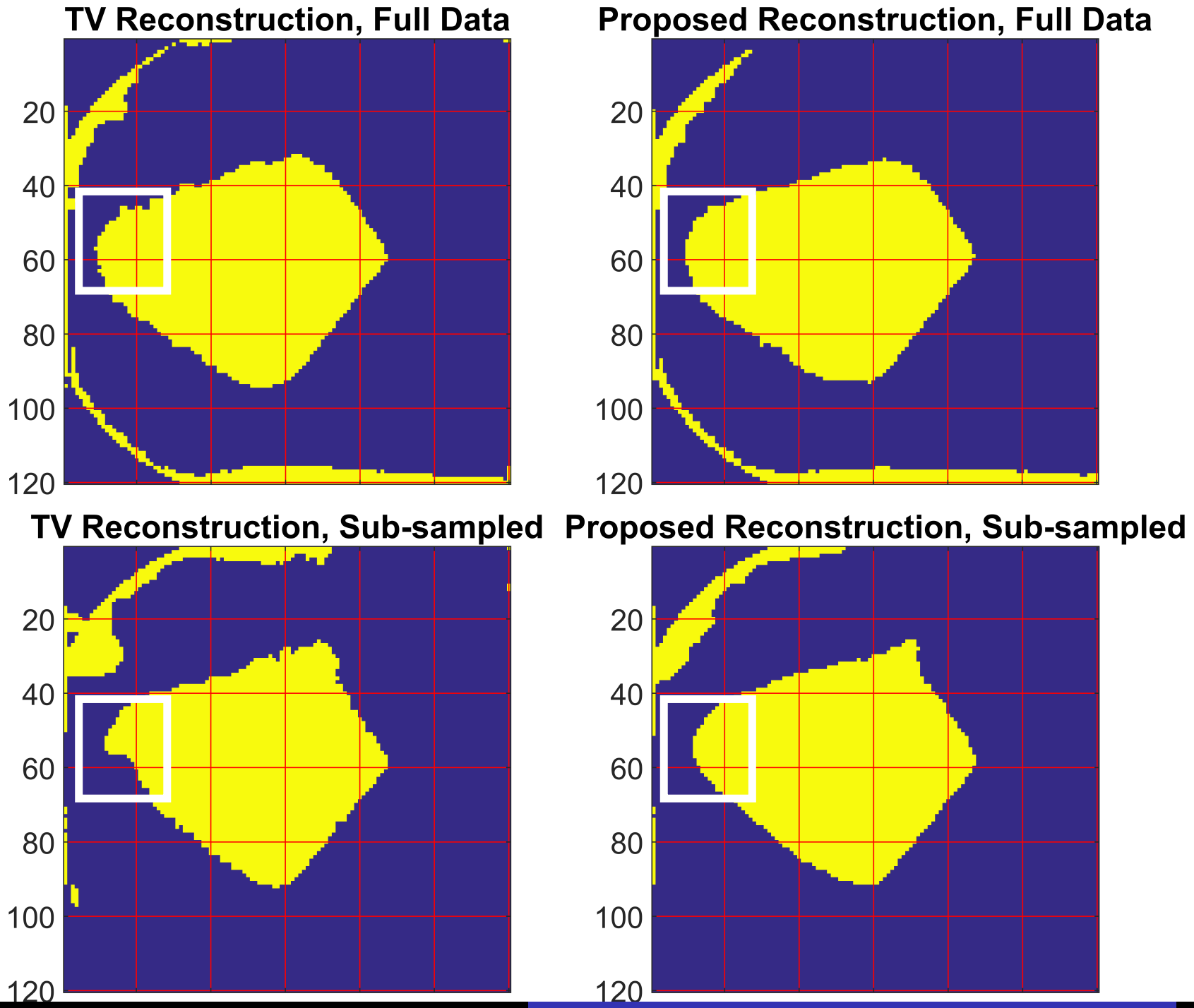
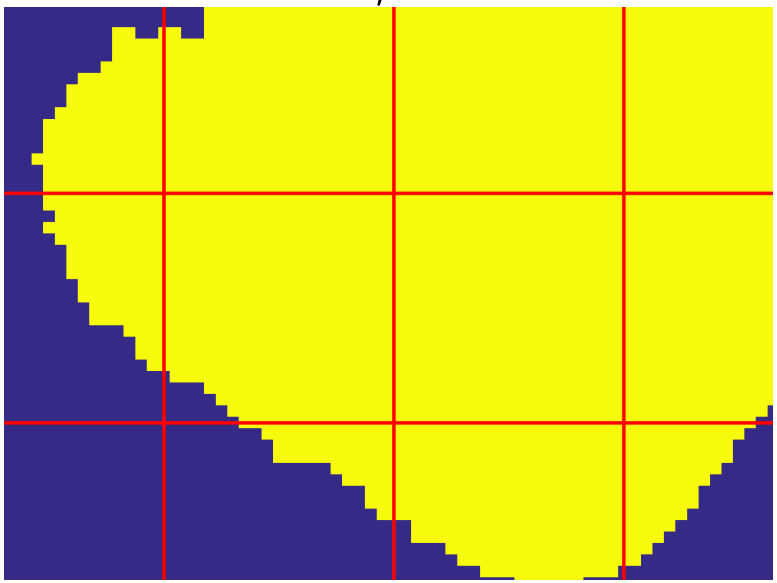
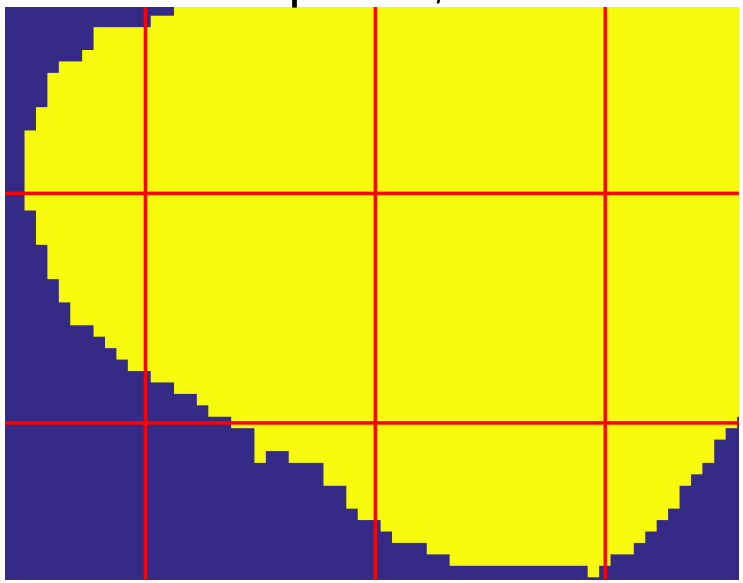



TV Reconstruction, Full Data
Proposed Reconstruction, Full Data
TV Reconstruction, Sub-sampled
Proposed Reconstruction, Sub-sampled
20
40
60
80
100
120

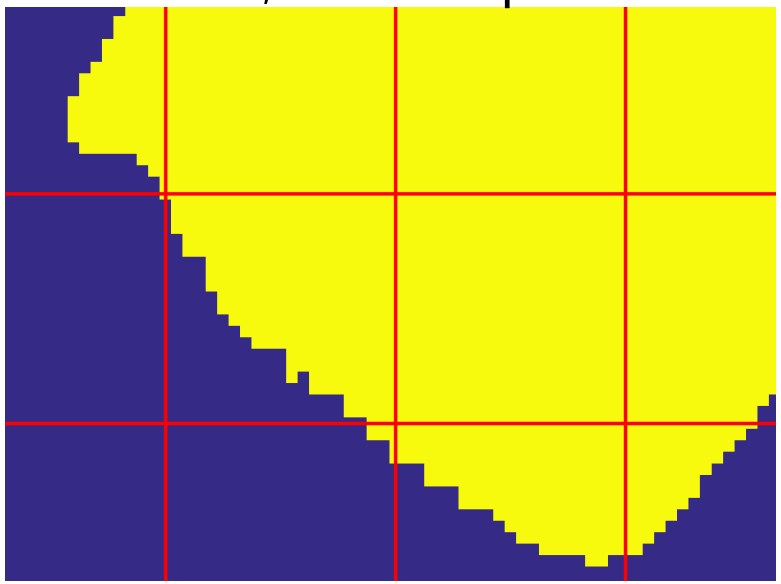
TV, full



Proposed, full



TV, sub-sampled



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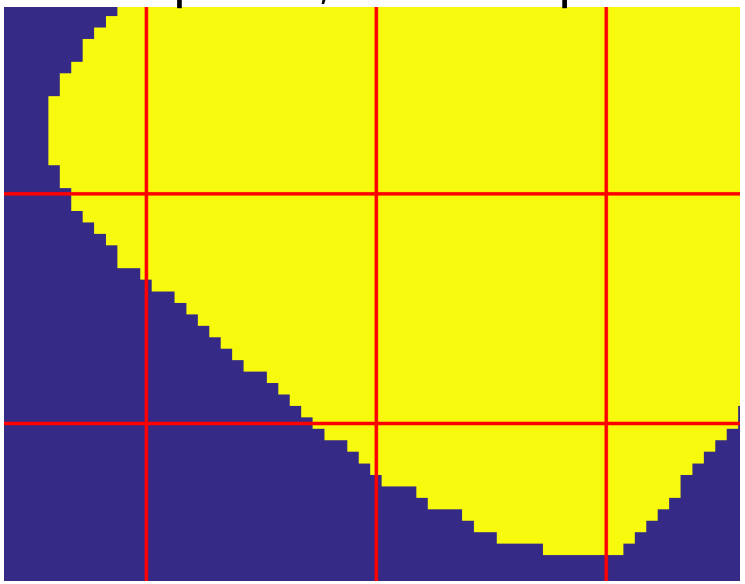

# Summary

- We have given an example where limited data is unavoidable
- Acknowledging the missing data explicitly allows us to mitigate errors
- Optimising where you are detecting structure on-the-fly is intrinsically hard
- We have given an example of the types of optimization tools available in this case
- A good choice of inpainting prior allows us to recover key geometrical features

# Thank you for your attention

For more information:
*Directional Sinogram Inpainting for Limited Angle Tomography*,
T., Benning, Brune, Lagerwerf, Collins, Leary, Midgley, Schönlieb
Inverse Problems 2019

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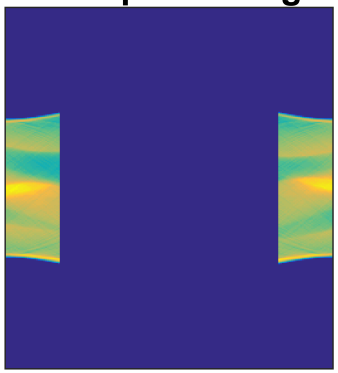
# Reconstructions from ‘bad’ Data

**Sub-sampled Sinogram**



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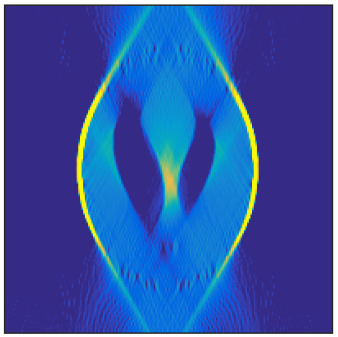

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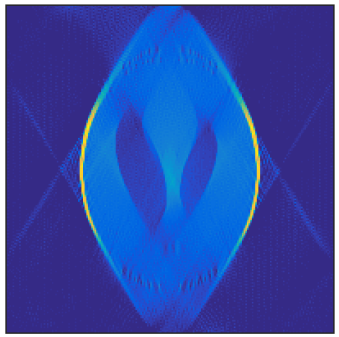
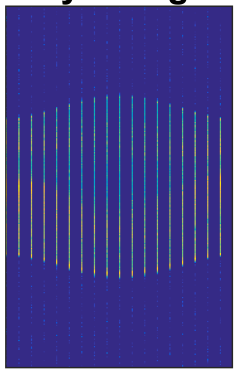


**TV reconstruction**

**Noisy Sinogram**



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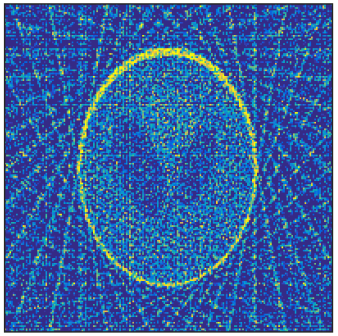

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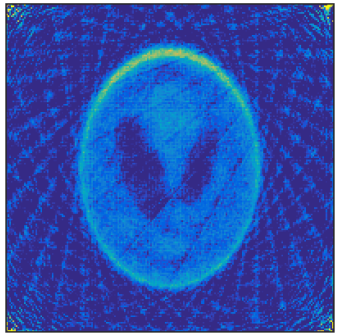

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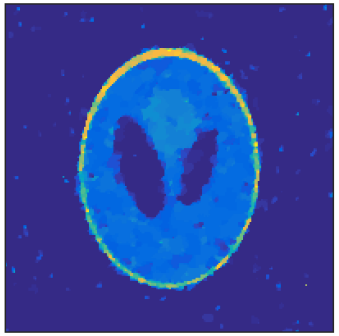

# Sketch Proof of Convergence

By construction of algorithm:

$$E(u_n, v_n) + \|v_n - v_{n-1}\|_2^2 \leq E(u_n, v) + \|v - v_{n-1}\|_2^2 \quad \forall v \qquad (*)$$

and equivalently for $u$.

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Summability:

$$(*) \implies \|u_n - u_{n-1}\|_2^2 + \|v_n - v_{n-1}\|_2^2 \leq E(u_{n-1}, v_{n-1}) - E(u_n, v_n)$$

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Summability:

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Limit points are critical points:

$$\begin{aligned}
(*) &\implies E(u_\infty, v_\infty) \leq E(u_\infty, v) + \|v - v_\infty\|_2^2 \\
&\implies \frac{E(u_\infty, v) - E(u_\infty, v_\infty)}{\|v - v_\infty\|} = O(\|v - v_\infty\|) \\
&\implies \partial_v E(u_\infty, v_\infty) = 0
\end{aligned}$$